International Journal of Theoretical Physics, Vol. 32, No. 7, 1993

WKB Approximation for a Quarkonium Equation

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Received December 6, 1992

A fourth-order differential equation recently proposed for describing quarkonia is studied. The eigenvalue spectrum is self-similar. A WKB approximation reproduces the spectrum and the so-called magic numbers which characterize the self-similarity.

1. DERIVATION

The common approach to ensure confinement for describing nonrelativistic quarkonia is a funnel-like potential. The underlying assumption is the validity of Schrödinger mechanics for quarkonia. However, other approaches are possible. Becker *et al.* (1991) sketch an ansatz which will be now developed in a simple way.

Consider the Schrödinger equation

$$[T+V-E]\,|\psi\rangle=0 \tag{1}$$

We modify this equation as simply as possible to build in confinement, but not by an additive potential.

The demand for simplicity obliges us to introduce further additive terms that admit only solutions describing bound states and never free or scattering states. Therefore the kinetic energy $\langle T\rangle$ should never be equal to the total energy E, i.e., $\langle T\rangle \neq E$. Free and scattering states should be suppressed by a term with denominator $\langle T\rangle - E$. If one writes the equation in this manner, the equation is not solvable exactly, but iteratively, because a solution $|\psi\rangle$ implies calculation of $\langle T\rangle$. But free states should

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0020-7748/93/0700-1225$07.00/0

not be possible if one uses the operator T instead of the expectation value $\langle T \rangle$, whatever this should mean.

If the denominator has the dimension of an energy, the numerator has the dimension of a squared energy. First one should introduce a mass scale η in the numerator, so one obtains the Schrödinger equation by switching off η, and otherwise η ensures confinement.

Therefore there must be another energy quantity in the numerator. The physical system—the quark–antiquark bound state—can be described by two quantities of dimension energy: the binding potential V and the mass scale η ensuring confinement. For generality we admit both possibilities. One does not known the mixture of the two terms. The general equation with built-in confinement is

$$\left[T + V + (A\eta)^2 \frac{1}{T-E} + (B\eta)\frac{V}{T-E} - E\right] |\psi\rangle = 0 \tag{2}$$

with mixture parameters A and B. Choosing $A^2 + B^2 = 1$ and $-B/A = \tan\varphi$ with phase φ, one obtains

$$\left[T + V + \frac{\eta^2 \cos^2\varphi}{T-E} - \frac{V\eta \sin\varphi}{T-E} - E\right] |\psi\rangle = 0 \tag{3}$$

To solve this quarkonium equation one reformulates it as

$$[(T-E)(T+V-E) - V\eta \sin\varphi + \eta^2 \cos^2\varphi]\,|\psi\rangle = 0 \tag{4}$$

or, with an additive function $|\chi\rangle$,

$$|\chi\rangle = \frac{\eta}{T-E-\eta \sin\varphi}\,|\psi\rangle \tag{5}$$

as

$$\begin{aligned} (T+V-E+\eta \sin\varphi)\,|\psi\rangle + \eta\,|\chi\rangle &= 0 \\ (T-E-\eta \sin\varphi)\,|\chi\rangle - \eta\,|\psi\rangle &= 0 \end{aligned} \tag{6}$$

Free or scattering states containing always a free part are explicitly inadmissible. The function $|\chi\rangle$ satisfies the equation

$$[(T+V-E)(T-E) - V\eta \sin\varphi + \eta^2 \cos^2\varphi]\,|\chi\rangle = 0 \tag{7}$$

It can be shown that the differential operator for $|\chi\rangle$ is adjoint to the operator for $|\psi\rangle$. Further interpretation can be found in Becker *et al.* (1991).

2. THEOREMS

For the quarkonium equation (3) one can easily derive the Hellmann–Feynman theorem and the following lemma in the norm $\langle\psi|\psi\rangle = 1$:

$$\frac{\partial E}{\partial\alpha} = \frac{1}{1-\langle\chi|\chi\rangle}\left[\left\langle\frac{\partial}{\partial\alpha}(T+V)\right\rangle_\psi + \langle\chi|\psi\rangle\frac{1}{\eta}\frac{\partial\eta^2}{\partial\alpha}\right.$$
$$\left. -\left\langle\frac{\eta^2}{(T-E-\eta\sin\varphi)^2}\frac{\partial T}{\partial\alpha}\right\rangle_\psi + (1+\langle\chi|\chi\rangle)\frac{\partial\eta\sin\varphi}{\partial\alpha}\right] \tag{8}$$

with the notation $\langle\cdot\rangle_\psi$ for $\langle\psi|\cdot|\psi\rangle$, etc.

A lemma in the norm $\langle\chi|\chi\rangle = 1$ can be calculated analogously. Note that for $\langle\psi|\psi\rangle = \langle\chi|\chi\rangle$ the derivation of the energy is singular for any parameter. For $\alpha = \varphi$ one obtains in the norm $\langle 1\rangle_\psi = 1$

$$\frac{\partial E}{\partial\varphi} = \frac{1+\langle\chi|\chi\rangle}{1-\langle\chi|\chi\rangle}\eta\cos\varphi$$
$$= \eta\cos\varphi + \mathcal{O}(\eta^3) \tag{9}$$

Integrating gives

$$E = E(\varphi=0) + \eta\sin\varphi + \mathcal{O}(\eta^3) \tag{10}$$

For $\alpha = \eta$

$$\frac{\partial E}{\partial\eta} = \frac{2\langle\psi|\chi\rangle + (1+\langle\chi|\chi\rangle)\sin\varphi}{1-\langle\chi|\chi\rangle} \tag{11}$$

and in $\langle 1\rangle_\chi = 1$

$$\frac{\partial E}{\partial\eta} = \frac{-2\langle\psi|\chi\rangle - (1+\langle\psi|\psi\rangle)\sin\varphi}{1-\langle\psi|\psi\rangle} \tag{12}$$

In the following we consider the equation with the Coulomb potential $V = -\lambda/r$ with coupling constant λ and distance r, reducing the number of variables by the introduction of dimensionless parameters (Quigg and Rosner, 1979). For m as reduced mass, i.e., double quark mass,

$$\varepsilon = \frac{E}{m\lambda^2}, \qquad \vartheta = \frac{\eta}{m\lambda^2}, \qquad \rho = rm\lambda \tag{13}$$

u as the radial function for ψ, v as the radial function for χ, and the abbreviations

$$\mathscr{L}_u = \varepsilon - \vartheta\sin\varphi - \frac{l(l+1)}{\rho^2} - v(\rho),$$
$$\mathscr{L}_v = \varepsilon + \vartheta\sin\varphi - \frac{l(l+1)}{\rho^2} \tag{14}$$

the quarkonium equation (6) takes the form

$$\begin{aligned} u'' &= -\mathscr{L}_u u + \vartheta v \\ v'' &= -\mathscr{L}_v v - \vartheta u \end{aligned} \tag{15}$$

Multiplying to the first equation $\rho^q u'$ with $q \geqslant -2l$, integrating this, and denoting the integrals

$$\langle F \rangle_f = \int_0^\infty d\rho\, F f^2, \qquad \langle F \rangle_{fg} = \int_0^\infty d\rho\, F f g \tag{16}$$

with f, g, and F as any functions, we obtain

$$\langle \rho^q \rangle_{u'u''} = -\langle \rho^q \mathscr{L}_u \rangle_{uu'} + \vartheta \langle \rho^q \rangle_{u'v} \tag{17}$$

Further calculation [parallel to Quigg and Rosner (1979)] turns the equation into

$$\begin{aligned} &[\rho^q u'^2 + \rho^q \mathscr{L}_u u^2 + 2\rho^q v v'' + \rho^q \mathscr{L}_v v^2 - q\rho^{q-1} u u' \\ &\quad + \tfrac{1}{2} q(q-1)\, \rho^{q-2} u^2 - \rho^q v'^2 + q\rho^{q-1} v v' - \tfrac{1}{2} q(q-1)\, \rho^{q-2} v^2]_{\rho=0} \\ &= -2q\langle \rho^{q-1} \mathscr{L}_u \rangle_u - \langle \rho^q \mathscr{L}'_u \rangle_u + 2q\langle \rho^{q-1} \mathscr{L}_v \rangle_v \\ &\quad - \langle \rho^q \mathscr{L}'_v \rangle_v + \langle \rho^{q-3} 4l(l+1) \rangle_v + 4q\vartheta \langle \rho^{q-1} \rangle_{uv} \\ &\quad - \tfrac{1}{2} q(q-1)(q-2)\langle \rho^{q-3} \rangle_u + \tfrac{1}{2} q(q-1)(q-2)\langle \rho^{q-3} \rangle_v \end{aligned} \tag{18}$$

This equation yields many theorems. For example, the virial theorem: Setting $q = 1$, we obtain

$$\langle T \rangle_\psi - \frac{1}{2} \left\langle r \frac{dV}{dr} \right\rangle_\psi = (E + \eta \sin \vartheta) \langle \chi | \chi \rangle + \eta \langle \chi | \psi \rangle \tag{19}$$

and therefore for potentials with $V(r) \sim r^k$

$$\langle T \rangle_\psi - \frac{k}{2} \langle V \rangle_\psi = (E + \eta \sin \varphi) \langle \chi | \chi \rangle + \eta \langle \chi | \psi \rangle \tag{20}$$

If we compare the expectation values of the kinetic energy using the virial theorem and the quarkonium equation, we find that the energy is given by

$$E = \frac{(1 + k/2)\langle V \rangle_\psi + \eta \sin \varphi (\langle 1 \rangle_\psi + \langle 1 \rangle_\chi) + 2\eta \langle \psi | \chi \rangle}{\langle 1 \rangle_\psi - \langle 1 \rangle_\chi} \tag{21}$$

The critical point is for $\langle\psi|\psi\rangle = \langle\chi|\chi\rangle$. If the energy is to be finite, the numerator must vanish. The critical mass scale is therefore

$$\eta_c = -\frac{(1+k/2)\langle V\rangle_\psi}{2(\langle\psi|\chi\rangle + \sin\varphi)} \tag{22}$$

For a Coulomb potential this equation reads

$$\eta_c = \frac{\lambda\langle r^{-1}\rangle_\psi}{4(\langle\psi|\chi\rangle + \sin\varphi)} \tag{23}$$

3. NUMERICAL RESULTS

Figure 1 shows the behavior of the ground state for variable phase φ and mass scale ϑ. For small ϑ one can recognize the sin φ dependence of the energy ε.

The self-similar spectrum first explored by R. Rosenfelder is shown in Fig. 2. Note that the states with $n = l+1$ are distinguished. A self-similar structure is built by the eigenvalues for constant l: Each eigenvalue function for a principal quantum number n is completely enveloped by every eigenvalue function with lower n.

Observing the S-states for $\varphi = 0°$, one can transform each function with $\theta = 4n^2\vartheta$ and $\zeta = 4n^2\varepsilon_n$. For higher n all curves are approximately equal.

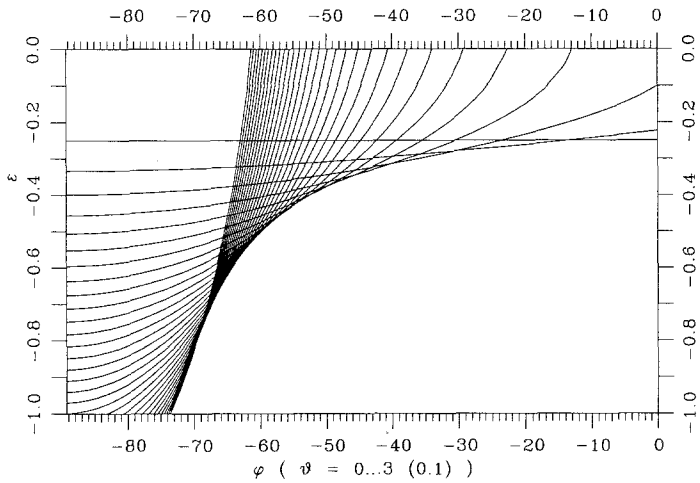

Fig. 1. Dimensionless energy ε as a function of phase φ and mass scale ϑ for the ground state. The curve for $\vartheta = 0$ is the straight line $\varepsilon = -0.25$.

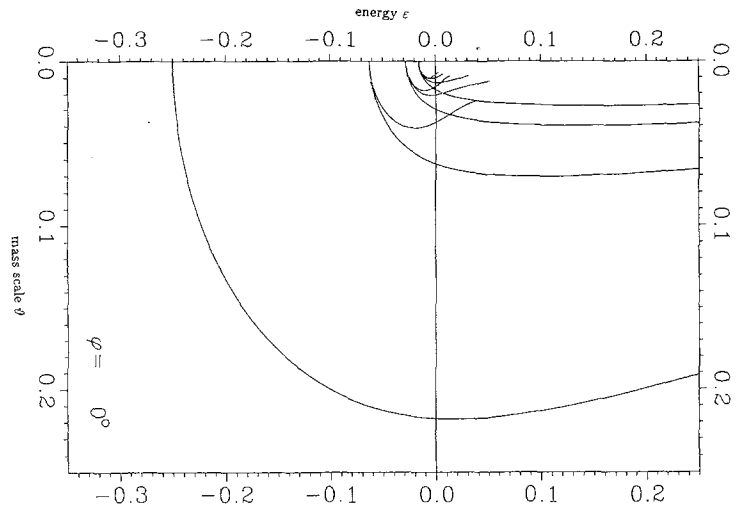

Fig. 2. Dimensionless energy ε as a function of mass scale ϑ at $\varphi = 0°$. For fixed n, the rightmost curve corresponds to the largest l value, $l = n - 1$.

The limiting value for increasing n should be $\theta_c \simeq 0.649$–0.650 and $\zeta_c \simeq 0.375$, evaluating the numerically determined values. The plot proving the self-similarity is shown in Fig. 3. This self-similarity holds practically for every n with the exception of the ground state, which has the critical mass scale $\theta_c \simeq 0.872$.

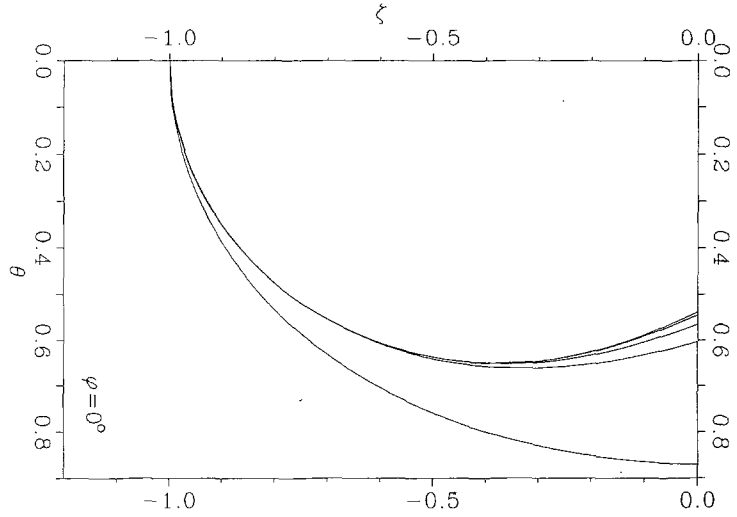

Fig. 3. Dimensionless energy ζ as a function of mass scale θ at $\varphi = 0°$ for fixed $l = 0$. The rightmost curve corresponds to the ground state, $n = 1$.

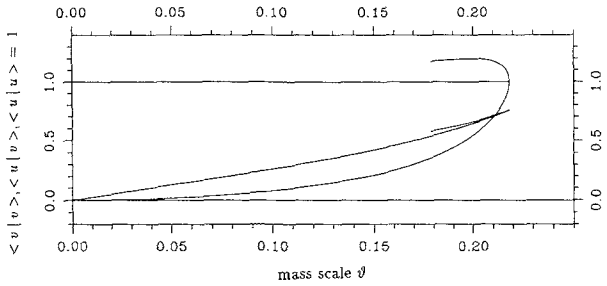

Fig. 4. Overlap integrals in the norm $\langle u|u\rangle = 1$ for the ground state as a function of mass scale ϑ at $\varphi = 0°$ for the ground state, $n = 1$, $l = 0$. The lowest curve for small ϑ corresponds to the integral $\langle v|v\rangle$.

Note that these values are dimensionless, determined only by the assumptions for integrability of the solution. Having calculated the curves for an infinite number of solutions, we can consider any enlargement of the spectrum and we would always see the same structure. Above the critical point the self-similarity is slowly switched off.

There exists a simple way to determine the critical point using the theorems developed above. Figure 4 illustrates an example. The point is critical if $\langle v|v\rangle = \langle u|u\rangle$, which is easy to realize for the numerically determined values.

4. WKB APPROXIMATION

For a WKB approximation we can use the quarkonium equation in u or the adjoint in v. It is easier to recognize which terms of the adjoint equation we have to give a power in $\hbar$. Therefore we take the adjoint for the equation with vanishing momentum $l = 0$ and Coulomb potential

$$0 = \left[\hbar^4 \frac{d^4}{d\rho^4} + \hbar^2(\rho^{-1} + 2\varepsilon)\frac{d^2}{d\rho^2} + \rho^{-1}(\varepsilon + \vartheta \sin\varphi) + \varepsilon^2 + \vartheta^2 \cos^2\varphi\right] v(\rho) \quad (24)$$

and the WKB Ansatz

$$v(\rho) = v_0 \sin\left(\frac{S(\rho)}{\hbar}\right) \quad (25)$$

In the dominant power we obtain

$$0 = S'^4 - (\rho^{-1} + 2\varepsilon)\, S'^2 + \rho^{-1}(\varepsilon + \vartheta \sin\varphi) + \varepsilon^2 + \vartheta^2 \cos^2\varphi \quad (26)$$

This equation of fourth order has the solutions

$$S' = \pm\left[\varepsilon + \frac{1}{2\rho} \pm \left(\frac{1}{4\rho^2} - \vartheta^2 \cos^2\varphi - \frac{\vartheta \sin\varphi}{\rho}\right)^{1/2}\right]^{1/2} \tag{27}$$

The classical substitute potential can be easily constructed from the quadratic equation in C for a quarkonium equation in classical mechanics following from

$$C = \frac{\eta^2 \cos^2\varphi}{T-E} - \frac{V\eta \sin\varphi}{T-E} \tag{28}$$

and

$$T - E = -(V + C) \tag{29}$$

Flügge (1971) and Migdal (1977) provide approaches in which the exact eigenvalue spectra are calculated for the Schrödinger Coulomb case. Migdal (1977) points out how to handle the case $l = 0$ without the trick of supplementing the equation with the term $(4\rho^2)^{-1}$ which must not be obvious for $l = 0$. But the problem is to give the right quantization condition for that approach. Migdal (1977) considers an asymptotic expression of the exact function for large n to derive the condition

$$\int_0^{-1/\varepsilon} d\rho\, S'(\rho) = n\pi \tag{30}$$

This quantization condition has the clear significance that the WKB function vanishes at $\rho = 0$ and at the turning point $\rho_w = -1/\varepsilon$. It is easy to prove that another quantization condition than the above does not give the exact Coulomb spectrum if n is identified as principal quantum number.

In the following we will make a WKB approximation for the case $\varepsilon < 0$ (which is the lower part of the self-similar spectrum) and $l \equiv \varphi \equiv 0$.

Solution of the turning point condition gives $\rho_w = -\varepsilon/(\varepsilon^2 + \vartheta^2)$ for ρ_w as turning point. The function S' is only real for $\varepsilon < -\vartheta$ for all $\rho \leqslant -\rho_w$. Choosing the quantization condition, we take the analogous condition to equation (30),

$$\int_0^{-\varepsilon/(\varepsilon^2+\vartheta^2)} d\rho\, S'(\rho) = n\pi \tag{31}$$

We do not prove this ansatz, but we motivate the following conditions:

1. The WKB function vanishes at both margin points $\rho = 0$ and $\rho = \rho_w$.

2. One of the remarkable results of the numerical solution is the self-similarity in the higher S-states. If we are to find this in the WKB approximation, we must choose the condition in this manner. Since the integral in equation (31) does not depend on the quantum number, but is a function of the parameters ε and ϑ, it gives only self-similarity in the transformation for the variables ζ and θ if (a) the integral is proportional to $1/(-\varepsilon)^{1/2}$ and a function F, and (b) this function F is only a function of the variable ϑ/ε.

If both conditions are satisfied, the WKB spectrum is self-similar.

We calculate the integral

$$\int_0^{-\varepsilon/(\varepsilon^2+\vartheta^2)} d\rho \left[\varepsilon+\frac{1}{2\rho}+\left(\frac{1}{4\rho^2}-\vartheta^2\right)^{1/2}\right]^{1/2} \equiv I \tag{32}$$

with $\varepsilon < -\vartheta \leqslant 0$. The transformation $x = 2\vartheta\rho$ and introduction of the variable $\Theta = \vartheta/|\varepsilon| = -\vartheta/\varepsilon$ gives

$$I = \frac{(-\varepsilon)^{1/2}}{2\vartheta} \int_0^{-2\vartheta\varepsilon/(\varepsilon^2+\vartheta^2)} d\rho \left[-1+\Theta\left(\frac{1}{x}+\left(\frac{1}{x^2}-1\right)^{1/2}\right)\right]^{1/2} \tag{33}$$

With a trick first found, so far as I know, by M. Stingl, we introduce the variable u,

$$u = \frac{1}{x} - \left(\frac{1}{x^2}-1\right)^{1/2} \tag{34}$$

and use the familiar properties

$$x = \frac{2u}{1+u^2}, \qquad \frac{1}{u} = \frac{1}{x} + \left(\frac{1}{x^2}-1\right)^{1/2} \tag{35}$$

Partial integration, partial fraction expansion

$$\frac{1}{1+y^4} = \frac{i}{2(y^2+i)} - \frac{i}{2(y^2-i)} \tag{36}$$

for $y = \sqrt{u}$, and use of the well-known primitive (Gröbner and Hofreiter, 1975)

$$\int dx \frac{1}{(c+x^2)(a-x^2)^{1/2}} = \frac{1}{\sqrt{c}\,(a+c)^{1/2}} \arctan \frac{x(a+c)^{1/2}}{(ca+cx^2)^{1/2}} \tag{37}$$

lead to the equation

$$2\Theta(-\varepsilon)^{1/2} I = i\Theta \left[\frac{1}{(-i\Theta-1)^{1/2}} \arctan\left(\frac{y(\Theta-i)^{1/2}}{(-i\Theta+iy^2)^{1/2}} \right) + \frac{1}{(i\Theta-1)^{1/2}} \arctan\left(\frac{y(\Theta+i)^{1/2}}{(i\Theta-iy^2)^{1/2}} \right) \right]_0^{\sqrt{\Theta}}$$

$$= \Theta \frac{[1+(1+\Theta^2)^{1/2}]^{1/2}}{\sqrt{2}\,(1+\Theta^2)^{1/2}} \pi \tag{38}$$

Evaluating the quantization condition gives

$$\varepsilon = -\frac{1}{8n^2} \frac{1}{1+\Theta^2} [1+(1+\Theta^2)^{1/2}] \tag{39}$$

The eigenvalues are self-similar and have the obvious property of reproducing the Schrödinger Coulomb values for $\Theta \to 0$. The equation can be properly applied only for $0 \leqslant \Theta \leqslant 1$ and negative energies. Since the error of a WKB approximation is of order $(n^2\pi^2)^{-1}$ (Migdal, 1977) a WKB approximation is only appropriate for large n.

Figure 5 illustrates the dimensionless parameter ε as a function of mass scale Θ. Comparing with the numerical values, we conclude that the

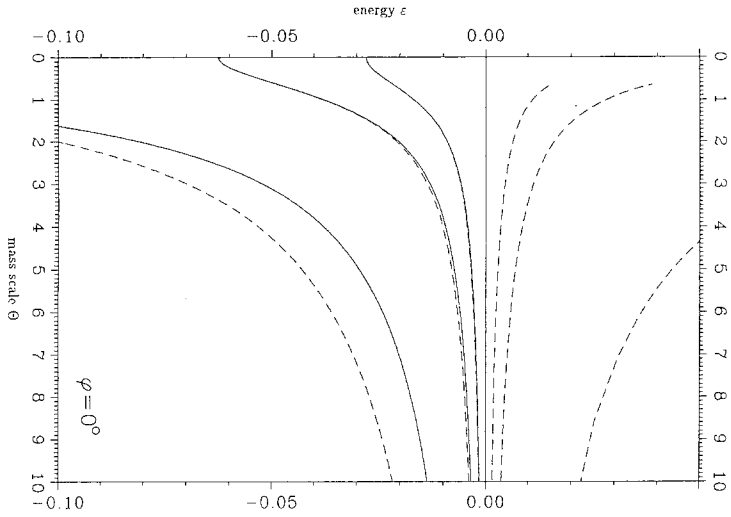

Fig. 5. WKB-approximate and numerically determined energy ε as a function of mass scale Θ at $\varphi = 0°$ for $l = 0$. The dashed lines correspond to the numerically determined values.

WKB function is a good approximation for $\Theta > 1$ with the exception of the ground state. For larger n the approximation is more appropriate.

It follows that

$$\vartheta \to \frac{1}{8n^2} \qquad \text{for} \quad \varepsilon \to 0, \quad \Theta \to \infty \tag{40}$$

We can expand for small ϑ,

$$\varepsilon = -\frac{1}{4n^2} + 3n^2\vartheta^2 + \dots \tag{41}$$

which can be found in a perturbation calculation of the exact quarkonium equation, too.

Since for the critical point Θ_c

$$\frac{\partial \varepsilon}{\partial \Theta} = \frac{-\varepsilon}{\Theta} \tag{42}$$

evaluation of the WKB formula gives

$$\Theta_c = \sqrt{3} \tag{43}$$

and therefore

$$\zeta_c = -\frac{3}{8} = -0.375 \tag{44}$$

$$\theta_c = \frac{3\sqrt{3}}{8} \simeq 0.6495191 \tag{45}$$

These values are exactly the numerically determined ones, although the WKB formula should not be applied at the critical point. But the magic number $\theta_c = (3\sqrt{3})/8$ characterizes the spectrum of the quarkonium Coulomb equation for the S-states.

5. CONCLUSIONS

The property of self-similarity is not restricted to chaotic behavior. The spectrum of a non-self-adjoint differential operator can have the self-similarity quality, too. But the structure has integer and nonfractal dimension. The self-similar structure can be simply characterized by magic numbers, similar to the well-known Feigenbaum structure.

The above approach has the advantage that the real spectrum of

quarkonia can be described by a model having a finite number of bound states and simultaneously no continuous spectrum, in contrast to a Schrödinger ansatz.

Further investigations should be made of the spectrum of the quarkonia equation with other than a Coulomb potential and in solving the problem of which quarkonium equation can describe quarkonia, i.e., which potential and which mass scale and phase should be chosen.

ACKNOWLEDGMENTS

This work was carried out while the author was with the Institut für Theoretische Physik I at the University of Münster, Germany. He thanks Prof. Dr. M. Stingl for his constructive comments.

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